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Geometrically finite groups acting on Busemann spaces

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ここでは, hyperbolic space に作用する群に対して定義される "geometrically finite" の概念を, Busemann space (CAT(0) space) に作用する群に拡張し, その性質を調べることを目的としている.

本稿で扱う空間はすべて proper geodesic space である. その定義は以下の通りである.

Definition 1. We say that a metric space (X, d) is a *geodesic space* if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such ξ is called a *geodesic*). Also a metric space (X, d) is said to be *proper* if every closed metric ball is compact.

まず, 3 種類の空間 "hyperbolic space", "CAT(0) space", "Busemann space" の定義を紹介する.

Definition 2. A proper geodesic space (X, d) is called a *hyperbolic space*, if there exists a number $\delta > 0$ such that every geodesic triangle in X is "δ-thin".

Here "δ-thin" is defined as follows: Let $x, y, z \in X$ and $\triangle := \triangle xyz$ a geodesic triangle in X. There exist unique non-negative numbers a, b, c such that

$$d(x, y) = a + b,\ d(y, z) = b + c,\ d(z, x) = c + a.$$

Then we can consider the metric tree $T_\triangle$ that has three vertexes of valence one, one vertex of valence three, and edges of length a, b and c. Let o be the vertex of valence three in $T_\triangle$ and let v_x, v_y, v_z be the vertexes of $T_\triangle$ such that $d(o, v_x) = a$,

$d(o, v_y) = b$ and $d(o, v_z) = c$. Then the map $\{x, y, z\} \to \{v_x, v_y, v_z\}$ extends uniquely to a map $f : \Delta \to T_\Delta$ whose restriction to each side of Δ is an isometry. For some $\delta \geq 0$, the geodesic triangle Δ is said to be *δ-thin*, if $d(p, q) \leq \delta$ for each points $p, q \in \Delta$ with $f(p) = f(q)$.

Definition 3. A proper geodesic space (X, d) is called a *CAT(0) space*, if the "CAT(0)-inequality" holds for all geodesic triangles Δ and for all choices of two points x and y in Δ.

Here the "CAT(0)-inequality" is defined as follows: Let Δ be a geodesic triangle in X. A *comparison triangle* for Δ is a geodesic triangle Δ' in the Euclidean plain $\mathbb{R}^2$ with same edge lengths as Δ. Choose two points x and y in Δ. Let x' and y' denote the corresponding points in Δ'. Then the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(x', y')$$

is called the *CAT(0)-inequality*, where $d_{\mathbb{R}^2}$ is the natural metric on $\mathbb{R}^2$.

Definition 4. A proper geodesic space (X, d) is called a *Busemann space*, if for each three points x_0, x_1, x_2 of X and each $t \in [0, 1]$,

$$d(\xi_1(td_1), \xi_2(td_2)) \leq td(x_1, x_2),$$

where $d_i = d(x_0, x_i)$ and $\xi_i : [0, d_i] \to X$ is a geodesic segment from x_0 to x_i for each $i = 1, 2$.

定義から直ちに CAT(0) space は Busemann space であることが分かる.

これらの hyperbolic space と Busemann space (そして CAT(0) space) は, boundary と呼ばれる空間を付け加えることによりコンパクト化できる. まずはこの boundary の集合としての定義を与える.

Definition 5. Let (X, d) be a hyperbolic space or Busemann space, and let $\mathcal{R}$ be the set of all geodesic rays in X. We define an equivalent relation $\sim$ in $\mathcal{R}$ as follows: For geodesic rays $\xi, \zeta : [0, \infty) \to X$,

$$\xi \sim \zeta \iff \operatorname{Im} \xi \subset B(\operatorname{Im} \zeta, N) \text{ for some } N \geq 0,$$

where $B(A, N) := \{x \in X \mid d(x, A) \leq N\}$. Then the *boundary* ∂X of X is defined as

$$\partial X = \mathcal{R}/\sim.$$

For each geodesic ray $\xi \in \mathcal{R}$, the equivalence class of ξ is denoted by $\xi(\infty)$.

X にこの boundary ∂X を付け加えた空間 $X \cup \partial X$ 上の位相を定義する前に, hyperbolic space と Busemann space の性質を紹介する.

Proposition 6 ([3], [4], [5]). *Let (X, d) be a hyperbolic space.*

(1) *For each $\alpha \in \partial X$ and each $x_0 \in X$, there exists a geodesic ray $\xi : [0, \infty) \to X$ such that $\xi(0) = x_0$ and $\xi(\infty) = \alpha$.*

(2) *For each $\alpha, \alpha' \in \partial X$ such that $\alpha \neq \alpha'$, there exists a geodesic line $\sigma : (-\infty, \infty) \to X$ such that $\sigma(\infty) = \alpha$ and $\sigma(-\infty) = \alpha'$.*

Proposition 7 ([9]). *Let (X, d) be a Busemann space.*

(1) *For each $\alpha \in \partial X$ and each $x_0 \in X$, there exists a unique geodesic ray $\xi : [0, \infty) \to X$ such that $\xi(0) = x_0$ and $\xi(\infty) = \alpha$.*

(2) *X is contractible.*

上述の Proposition 6 (2) は一般に Busemann space では成り立たず, また, Proposition 7 (2) は一般に hyperbolic space では成り立たない.

Proposition 6 (1) と Proposition 7 (1) の性質を用いて $X \cup \partial X$ 上の位相を以下のように定義することができる.

Definition 8 ([3], [4], [9]). Let (X, d) be a hyperbolic space or Busemann space, and let $x_0 \in X$. We define a topology on $X \cup \partial X$ by the following conditions:

(1) X is an open subspace of $X \cup \partial X$.

(2) Let $\alpha \in \partial X$, $r > 0$ and $\epsilon > 0$. Then there exists a geodesic ray ξ such that $\xi(0) = x_0$ and $\xi(\infty) = \alpha$ by Proposition 6 (1) or Proposition 7 (1). Let

$$U_{x_0}(\alpha; r, \epsilon) = \{x \in X \cup \partial X \mid x \notin B(x_0, r),\ d(\xi(r), \xi_x(r)) < \epsilon\},$$

where $\xi_x : [0, d(x_0, x)] \to X$ is a geodesic (segment or ray) from x_0 to x. Let $\epsilon_0 > 0$ be a constant such that if X is hyperbolic then $\epsilon_0 > 2\delta$ (where δ is the number in Definition 2). Then the set

$$\{U_{x_0}(\alpha; r, \epsilon_0) \mid r \in \mathbb{N}\}$$

is a neighborhood basis for α in $X \cup \partial X$.

この $X \cup \partial X$ 上の位相の定義は X の点 x_0 を用いて与えられているが, 実際にはこの位相が x_0 の取り方に拠らず, また, $X \cup \partial X$ が compact metrizable space となることが知られている ([3], [4], [5], [9]).

いま，hyperbolic space または Busemann space 上に，properly discontinuous に isometry として作用する群を考える．ここで properly discontinuous の定義は次の通りである．

Definition 9. An action of a group Γ on a metric space (X, d) is said to be *properly discontinuous*, if the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite for every compact subset K of X.

Hyperbolic space または Busemann space 上に properly discontinuous に isometry として作用する群に対して，"群の limit set" が以下で定義される．

Definition 10. Let (X, d) be a hyperbolic or Busemann space, and let Γ be a group which acts properly discontinuously by isometries on X. The *limit set of* Γ (*with respect to* X) is defined as

$$L(\Gamma) = \partial X \cap \mathrm{cl}_{X \cup \partial X} \Gamma x_0,$$

where $\mathrm{cl}_{X \cup \partial X}$ means the closure in $X \cup \partial X$, and x_0 is a point in X. We note that the limit set $L(\Gamma)$ is independent of a point $x_0 \in X$.

ここで，hyperbolic space 上に properly discontinuous に isometry として作用する群に対して，"geometrically finite" が次で定義される．

Definition 11 ([10]). Let (X, d) be a hyperbolic space and Γ a group which acts properly discontinuously by isometries on X. We say that (the action of) Γ is *geometrically finite* (*with respect to* X), if there exists a compact subset K of X such that $\mathcal{L}(L(\Gamma)) \subset \Gamma K$, where $\mathcal{L}(L(\Gamma))$ is the union of the images of all geodesic lines $\sigma : (-\infty, \infty) \to X$ such that $\sigma(-\infty), \sigma(\infty) \in L(\Gamma)$.

この geometrically finite group に関して，次の定理が知られている．

Theorem 12 (Ranjbar-Motlagh [10]). *Let X be a hyperbolic space and Γ a group which acts properly discontinuously by isometries on X.*

(i) *Suppose that $H \subset G$ are two subgroups of Γ and H is geometrically finite. Then, $L(G) = L(H)$ if and only if $[G : H] < \infty$.*

(ii) *Let G be a subgroup of finite index in Γ. Then Γ is geometrically finite if and only if G is geometrically finite.*

(iii) *If G_1 and G_2 are two geometrically finite subgroups of Γ, then $G_1 \cap G_2$ is also geometrically finite and $L(G_1 \cap G_2) = L(G_1) \cap L(G_2)$.*

ここでの目的は，この geometrically finite の概念を Busemann space に作用する群に拡張することである．Hyperbolic space に作用する群に対して Definition 11 が有効であるのは，geodesic line に関する Proposition 6 (2) の性質が hyperbolic space では成り立つためである．この geodesic line の性質が一般には成り立たない Busemann space で "geometrically finite" を定義するために，Definition 11 の中の "geodesic line" を "geodesic ray" で置き換えることを試みた．そして実際，次の命題を得ることができた．

Proposition 13. *Let (X,d) be a hyperbolic space and Γ a group which acts properly discontinuously by isometries on X. Then the following statements are equivalent:*

(1) *The action of Γ is geometrically finite.*
(2) *There exists a compact subset K of X such that $\mathcal{R}_{x_0}(L(\Gamma)) \subset \Gamma K$ for some $x_0 \in X$, where $\mathcal{R}_{x_0}(L(\Gamma))$ is the union of the images of all geodesic rays ξ issuing from x_0 with $\xi(\infty) \in L(\Gamma)$.*

上述の結果をもとに，Busemann space 上に作用する群に対して "geometrically finite" を次で定義する．

Definition 14. Let (X,d) be a Busemann space and Γ a group which acts properly discontinuously by isometries on X. We say that (the action of) Γ is *geometrically finite* (*with respect to X*), if there exists a compact subset K of X such that $\mathcal{R}_{x_0}(L(\Gamma)) \subset \Gamma K$ for some $x_0 \in X$.

Hyperbolic でありかつ Busemann である空間上に作用している群に関して，Definition 11 と Definition 14 の二つの "geometrically finite" の概念が一致していることを Proposition 13 は保証している．

この定義のもと，Theorem 12 が Busemann space で成り立つことを証明した．

Theorem 15. *Let X be a Busemann space and Γ a group which acts properly discontinuously by isometries on X.*

(i) *Suppose that $H \subset G$ are two subgroups of Γ and H is geometrically finite. Then, $L(G) = L(H)$ if and only if $[G : H] < \infty$.*
(ii) *Let G be a subgroup of finite index in Γ. Then Γ is geometrically finite if and only if G is geometrically finite.*

(iii) *If G_1 and G_2 are two geometrically finite subgroups of Γ, then $G_1 \cap G_2$ is also geometrically finite and $L(G_1 \cap G_2) = L(G_1) \cap L(G_2)$.*

Hyperbolic space に作用する geometrically finite group に関する最近の話題としては, E. L. Swenson によって "geometrically finite group" と "quasi-convex group" が一致する概念であることが [12] の中で最近証明されている. Busemann space においてもこのことが成り立つのかどうかについて今後研究を行いたい.

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